1. Let \( A \) be a matrix. Find
\[
A = \begin{bmatrix}
5 & -6 & -6 \\
-1 & 4 & 2 \\
3 & -6 & -4
\end{bmatrix}
\]
(a). the characteristic polynomial and the minimal polynomial of \( A \). (6分)
(b). the eigenvalues and eigenspaces of \( A \). (6分)
(c). an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonal and use it to find \( A^{10} \). (8分)

2. Let \( A \) be an \( m \times n \) real matrix, \( B \) be an \( n \times p \) real matrix.
Prove that \( \text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n \). (20分)

3. Let \( P \in \mathbb{R}^{n \times n} \) be nonsingular and \( A \in \mathbb{R}^{m \times n} \). Prove that the column space of \( AP \) is equal to the column space of \( A \). In particular, \( AP \) and \( A \) have the same column rank. (15分)

4. Let \( A \) and \( B \) be two \( n \times n \) matrices. Show that \( (AB-I) \) is invertible if \( (BA-I) \) is invertible, where \( I \) is an \( n \times n \) matrix. (15分)

5. Let \( A, B \in \mathbb{R}^{n \times n} \) be nonzero matrices. Prove that
(a). if \( A \) and \( B \) are similar, then they have the same eigenvalues. (4分)
(b). if \( A \) is diagonalizable and its eigenvalues are all \( \pm 1 \), then \( A = A^{-1} \). (4分)
(c). if \( A \) is nilpotent, i.e., \( A^k = 0 \), \( \exists k \in \mathbb{N} \), then all of its eigenvalues are equal to zero. (3分)
(d). if \( A \) is nilpotent, then \( A \) is not diagonalizable. (4分)

6. Find the value \( c \) so that the system of linear equations
\[
\begin{align*}
x + y + z &= 1 \\
x - y + z &= 2 \\
x + y - z &= c
\end{align*}
\]
has solutions in \( \mathbb{R}^3 \), and in that case, find all the solutions. (15分)