國立嘉義大學 99 學年度
應用數學系碩士班（甲組）招生考試試題

科目：線性代數

說明：(1) 本考試試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。
(2) 第 1~4 題每題 10 分，第 5~8 題每題 15 分，共 100 分。

1. Fill in the six entries: \(a, b, c, d, e, f\) in the following \(4 \times 4\) matrix

\[
\begin{bmatrix}
1 & -1 & a & 5 \\
b & 4 & c & 8 \\
2 & -7 & -1 & d \\
e & f & 6 & 3
\end{bmatrix}
\]

so that the matrix is symmetric. (10%)

2. Let \(A\) and \(B\) be two square matrices. Does

\[(A + B)^2 = A^2 + 2AB + B^2\]

hold? If so, prove it; if not, give a counterexample and state under what conditions the equation is true. (10%)

3. Let \(A\) be a \(m \times n\) real matrix. Prove \(A^tA\) is nonsingular iff \(A\) has linearly independent columns. (10%)

4. Let \(A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\). Find an orthogonal basis for the column space of \(A\). (10%)

5. For vectors \(v\) and \(w\) in an inner product space. Prove that \(v - w\) and \(v + w\) are perpendicular if and only if \(\|v\| = \|w\|\). (15%)

6. Find a diagonal 3 by 3 matrix \(D\) and an invertible 3 by 3 matrix \(P\) such that

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} = P^tDP. \quad (15%)
\]

7. Compute the inverse of

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

by Gauss-Jordan elimination. (15%)

8. Let \(V, W\) be vector spaces, and suppose that \(\{v_i\}_{i=1}^n\) is a basis for \(V\). Prove that: for any \(\{w_i\}_{i=1}^n \subset W\), there exists exactly one linear transformation \(f : V \to W\) such that \(f(v_i) = w_i\) for each \(i \in \{1, \cdots, n\}\). (15%)